TITLE: Systems of Delay Equations with Small Solutions: A Numerical Approach

Systems of delay equations with small solutions: a numerical approach

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Abstract

We consider systems of delay differential equations of the form

$$y'(t) = A(t)y(t-1)$$

where $y \in \mathbb{R}^n$ and $A : \mathbb{R} \to \mathbb{R}^{n\times n}$. We investigate whether a numerical method can be used to determine whether or not the equation has so-called *small* solutions. Our work builds on recent analysis and experimental work completed in the scalar case and we are able to conclude that, at least when A is a suitable periodic matrix, one can predict small solutions by using a numerical approximation scheme of fixed step length.

1 Introduction and basic theory

The analysis of delay differential equations, both analytically and numerically, is well-established. One distinctive feature is that even a scalar delay differential equation is an infinite dimensional problem. For, if x satisfies

$$y'(t) = b(t)y(t-1) \tag{1.1}$$

the initial conditions that need to be specified take the form

$$y(t) = \varphi(t), \quad -1 \leq t \leq 0. \tag{1.2}$$

This infinite dimensionality has two significant implications for us:

(1) the dimension of a system of delay equations is *the same* as the dimension of a scalar delay equation, and

(2) the range of dynamical behaviour among solutions of delay equations is far wider than would be the case for ordinary differential equations.

In the present paper we are investigating an infinite dimensional property (that of possessing small solutions) where the analysis and results for systems needs to be presented quite separately from those for scalar equations because there are some interesting and distinctive features.

One way in which delay equations may be analysed is to view the solution operator as a dynamical system. The dimension of the dynamical system then inherits the infinite dimensionality of the delay equation itself. Small solutions (those that satisfy $x(t)e^{\alpha t} \to 0$

as $t \to \infty$ for all values of the parameter α) can arise in these infinite dimensional problems but would not be observed in finite dimensional equations. They are important because, when a delay equation has small solutions, the eigenfunctions and generalised eigenfunctions of the solution map do not form a complete set. This means that some standard analytical results do not hold and that particular care must be taken in solving and analysing the equation.

The easy detection of problems that have small solutions is still, in general, open, but we have seen [4, 5] that the use of a numerical approximation scheme can lead to good insights. Here we approximate the delay differential equation using a simple numerical scheme with fixed step length and then consider the spectrum of the resulting solution map.

In recent work (see, for example [3, 5]) the scalar case has been considered with some success. We have been able to see that, for the equation (1.1) with b periodic of period 1, we can detect the existence of small solutions by exploring the (finitely many) eigenvalues of the numerical scheme. We also found that it was not necessary to use a sophisticated numerical scheme for the investigation and this has justified us in focussing on the trapezium rule as the numerical method in this paper.

For the scalar case (1.1) it is known (see for example [4, 5]) that, when b satisfies the periodicity condition $b(t) = b(t-1)$, then non-trivial small solutions arise if and only if the function b changes sign. For the vector-valued case we can give a theorem, recently proved by Verduyn Lunel ([11]).

Theorem 1.1 *Consider the equation*

$$y'(t) = A(t)y(t-1), \text{ where } A(t) = A(t-1), \tag{1.3}$$

and where $y \in I\!R^n$. The equation has small solutions if and only if at least one of the eigenvalues λ_i satisfies, for some $\hat{t}$,

$$\Re\lambda_i(\hat{t}-) \times \Re\lambda_i(\hat{t}+) < 0, \lambda_i(\hat{t}) = 0. \tag{1.4}$$

Remark 1.2 *We shall describe the property (1.4) using the words an eigenvalue passes through the origin. We note that, even for real matrices A, the eigenvalues may be complex and it could be that a pair of complex conjugate eigenvalues will cross the $y-$axis away from the origin. In this case the equation has small solutions only if there is some other crossing of the $y-$axis by an eigenvalue where the crossing does take place at the origin.*

2 Numerical methods and systems of order two

All the important relevant features of systems of delay equations turn out to be exhibited in systems of two equations and so we shall focus on these for simplicity. We consider the equation

$$y'(t) = A(t)y(t-1) \quad \text{for} \quad A \in \mathbb{R}^{2\times 2} \quad \text{and} \quad y \in \mathbb{R}^2. \tag{2.1}$$

subject to $y(t) = \varphi(t)$ for $-1 \le t \le 0$ and we assume that $A(t) = A(t-1)$ for all t.

We introduce

$$y(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}, \quad \varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}. \tag{2.2}$$

We apply the trapezium rule with step length $h = \frac{1}{N}$ and introduce the approximations $x_{1,j} \approx x_1(jh)$, and $x_{2,j} \approx x_2(jh), j > 0$; $x_{1,j} = \varphi_1(jh), x_{2,j} = \varphi_2(jh), -N \le j \le 0$. Set

$$y_n = \left(\; x_{1,n}, x_{1,n-1}, \dots, x_{1,n-N}, x_{2,n}, x_{2,n-1}, \dots, x_{2,n-N} \; \right)^T. \tag{2.3}$$

We note that, as in the one-dimensional case (see [3, 4, 5]), we can write the numerical scheme as $y_{n+1} = A(n)y_n$, where the matrix $A(n)$ now takes the form

$$A(n) = \begin{pmatrix}
1 & 0 & \dots & 0 & \frac{h}{2}\alpha_{n+1} & \frac{h}{2}\alpha_n & 0 & \dots & \dots & 0 & \frac{h}{2}\beta_{n+1} & \frac{h}{2}\beta_n \\
1 & 0 & \dots & \dots & \dots & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\
0 & 1 & \ddots & & & \vdots & \vdots & & & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & & & & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & & & & & \vdots \\
0 & \dots & \dots & 0 & 1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\
0 & \dots & \dots & 0 & \frac{h}{2}\gamma_{n+1} & \frac{h}{2}\gamma_n & 1 & 0 & \dots & 0 & \frac{h}{2}\delta_{n+1} & \frac{h}{2}\delta_n \\
0 & \dots & \dots & \dots & \dots & 0 & 1 & 0 & \dots & \dots & \dots & 0 \\
\vdots & & & & & \vdots & 0 & 1 & \ddots & & & \vdots \\
\vdots & & & & & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & & & & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & \dots & \dots & \dots & 0 & 0 & \dots & \dots & 0 & 1 & 0
\end{pmatrix}. \tag{2.4}$$

The sequence of matrices $\{A(n)\}$ is periodic, of period N (since the function A is periodic of period 1) and $y_2 = A(1)y_1, y_3 = A(2)A(1)y_1$ and so on. Therefore $y_{N+1} = Cy_1$ where $C = A(N)A(N-1).\cdots.A(2)A(1)$.

Remark 2.1 *The key to extending our discussion to larger systems, and indeed, to gaining a full understanding of the approach, is to note that in both the matrix $A(n)$ and the matrix C the original block structure is retained. Therefore although the matrices $A(n)$ and C are considerably larger than the original 2×2 matrix $A(t)$ in the problem, they are made up of 4 blocks in a 2×2 formation. Indeed the contents of each block is completely determined by our numerical method (the trapezium rule) and the values of the corresponding function, respectively $\alpha, \beta, \gamma, \delta$. There is no pollution of the blocks from the neighbouring functions.*

We consider three different cases:

(1) $\beta(t) = \gamma(t) = 0$ so that the matrix A is diagonal,

(2) either $\beta(t) = 0$ or $\gamma(t) = 0$ so that the matrix A is triangular, and

(3) the matrix A is neither diagonal nor triangular.

The first two cases can be dealt with quite quickly because of the fact that real diagonal and triangular matrices have only real eigenvalues and these eigenvalues lie on the diagonal. Therefore in these two cases we need consider only the question of whether the eigenvalues pass through zero; we do not need to concern ourselves with possible complex eigenvalues whose real parts change sign away from the origin.

We can go further: a diagonal matrix A leads to a block diagonal matrix $A(n)$ (with non-zero blocks top left and bottom right). Now by simple matrix theory we know that the eigenvalues of such a matrix are simply the union of the eigenvalues of the two blocks. A similar argument applies when there is a triangular matrix A because the matrices $A(n)$ are then block triangular. It follows that, for both of cases 1 and 2, the 2−dimensional eigenvalue problem simply reduces to two 1−dimensional problems. Therefore, when we consider the eigenspectra of the numerical schemes in cases 1 and 2, we expect the result to be the superposition of the eigenspectra from the two block matrices on the diagonal of C.

Case 3 is more complicated and we shall return to it after we give brief examples of Cases 1 and 2.

3 How to recognise small solutions: our previous work

Space restrictions here prevent us from giving a great many details of our previous work, but we provide a summary to show how the current investigation builds on the scalar case. In [3] we considered the eigenspectra of the matrix C. We showed that there were three characteristic patterns for the eigenspectra, represented by Figure 1. We take the presence of the closed loops that cross the x-axis to be characteristic of the cases where small solutions arise.

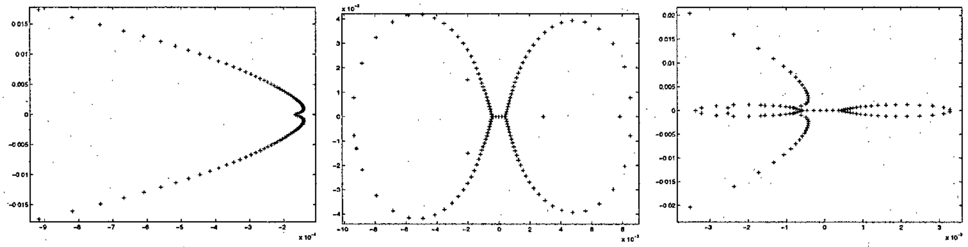

FIG. 1. Eigenspectra where $b(t)$ has no change of sign on $[0,1]$ (left), where $b(t)$ has a change of sign on $[0,1]$ and $\int_0^1 b(s)ds = 0$ (centre), and where $b(t)$ has a change of sign on $[0,1]$ and $\int_0^1 b(s)ds \neq 0$ (right).

4 The cases when $\beta(t) = 0$ and/or $\gamma(t) = 0$

As we have remarked already, the eigenspectrum when A is diagonal or triangular is just the same as the eigenspectra of the block matrices from the diagonal of C. We expect to

find the eigenspectra superimposed, which is indeed what we see in the examples given. Here we assume that at least one of $\gamma(t)$ or $\beta(t)$ is zero; the plots are then independent of the values taken by the other.

Example 4.1 We solve (2.1) with the choice $\alpha(t) = \sin 2\pi t + 1.4$ and $\delta(t) = \sin 2\pi t + 0.5$. Here α does not change sign but δ does change sign. We expect small solutions and Figure 2 provides confirmation.

Example 4.2 Now we solve (2.1) with $\alpha(t) = \sin 2\pi t$ and $\delta(t) = \begin{cases} -0.3 & \text{for } t \in (0, \frac{1}{2}], \\ 0.7 & \text{for } t \in (\frac{1}{2}, 1]. \end{cases}$ This time both α and δ change sign and we expect small solutions (see Figure 2).

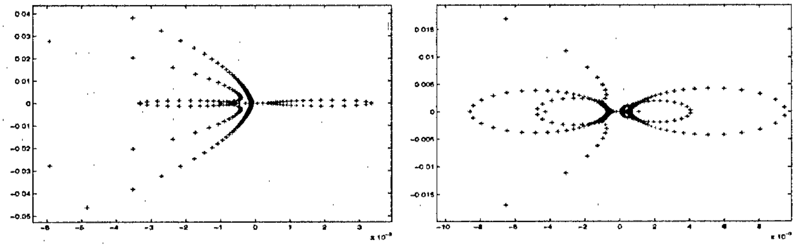

FIG. 2. Eigenspectra for Example 4.1 (left) and Example 4.2 (right).

4.1 The general two dimensional case

We now move on to consider the case when neither of $\beta(t), \gamma(t)$ is identically zero. In this situation the eigenvalues of $A(t)$ can be complex and so may cross the y-axis away from the origin.

First, we recall that $\det(A)$ is the product of the eigenvalues of A so that, by Theorem 1.1, it follows that $\det(A) = 0$ is a necessary condition for small solutions. However this condition cannot be used to characterise equations where small solutions arise; if the eigenvalues of A are real and one passes through the origin, then $\det(A)$ will change sign. If the eigenvalues of A are a complex conjugate pair and cross the y-axis at the origin then $\det(A)$ will instantaneously take the value zero but will otherwise remain positive (the same behaviour as when a real eigenvalue becomes zero but does not change sign). Therefore one cannot expect a change of sign in $\det(A)$ whenever there are small solutions. The fact that the trace of A is the sum of the eigenvalues of A can be used to characterise this case.

We summarise. For a real matrix A:

(1) if $\det(A)$ changes sign then there are small solutions,
(2) if $\det(A)$ becomes zero instantaneously and $\mathrm{trace}(A)$ simultaneously changes sign then there are small solutions,
(3) if $\det(A)$ becomes zero instantaneously and $\mathrm{trace}(A)$ does not simultaneously change sign then there are no small solutions indicated.

Example 4.3 We first consider the case when the matrix A takes the form

$$A(t) = \begin{pmatrix} \sin 2\pi t + a & \sin 2\pi t + b \\ \sin 2\pi t + c & \sin 2\pi t + d \end{pmatrix}.$$

By judicious choice of the constants a, b, c, d one can produce different types of behaviour. One can see that $|A(t)| = (a + d - b - c)\sin 2\pi t + (ad - bc)$. We will illustrate with the following choices of the constants

Case 1: $a = 1.5, b = 0.7, c = 0.5, d = 0.5$ where the determinant changes sign,
Case 2: $a = -2, b = 0.8, c = 1.8, d = 0.7$ where, again, the determinant changes sign,
Case 3: $a = 1.6, b = 0.8, c = 1.8, d = 0.7$ where the determinant never becomes zero.

From the plots for cases 1 and 2, we can easily see the presence of small solutions in the eigenspectra shown in Figure 3. In the Case 3, the eigenspectra in Figure 3 indicate that, as expected, no small solutions are present.

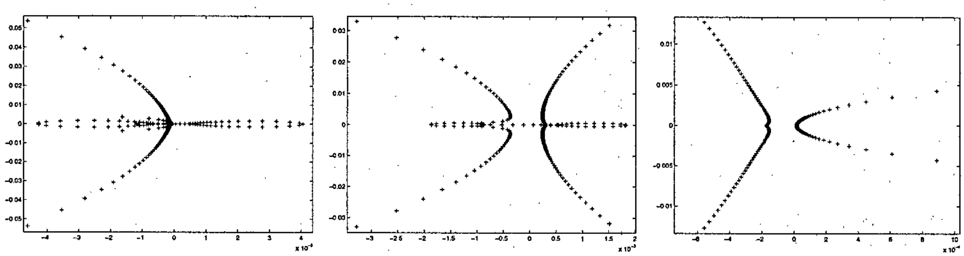

FIG. 3. Case 1. Case 2. Case 3

Example 4.4 Next, we consider the case when the matrix A takes the form

$$A(t) = \begin{pmatrix} \sin 2\pi t & -(\sin 2\pi t + b) \\ \sin 2\pi t + b & \sin 2\pi t \end{pmatrix}.$$

We choose the constant b in the following ways

Case 4: $b = 0$ so that $\det(A)$ becomes instantaneously zero at the same value that trace(A) changes sign and the complex eigenvalues of A cross the y-axis at the origin,

Case 5: $b = 0.05$ so that the complex eigenvalues of A cross the y-axis away from the origin.

Here we can see that the characteristic shapes we familiar from our earlier work are not reproduced and further investigation is called for. We remark that (in the zoomed versions) the eigenspectrum where small solutions arise passes through the origin. This property is reproduced also for all other examples that we have tried.

Example 4.5 Now we consider the case when the matrix A takes the form

$$A(t) = \begin{pmatrix} t & t + b \\ -t - b & t \end{pmatrix}$$

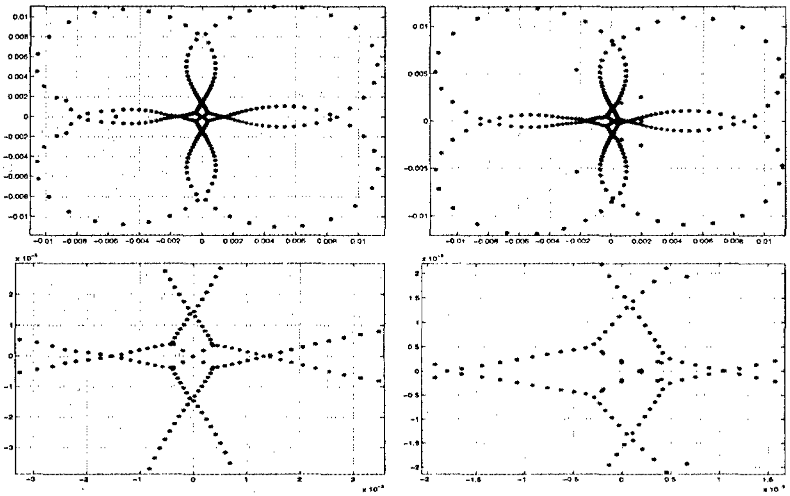

FIG. 4. Left: Case 4. Right: Case 5 and (below) zoomed versions.

for $t \in [-0.5, 0.5), A(t) = A(t-1)$ for $t \geq 0.5$ then it follows that A has complex eigenvalues that cross the $y-$axis at $y = b$ when $t = 0$. We plot the eigenspectra for

Case 6: $b = 0$ so the eigenvalues of A cross the y-axis at the origin,

Case 7: $b = 0.01$ so the eigenvalues of A cross the y-axis away from the origin.

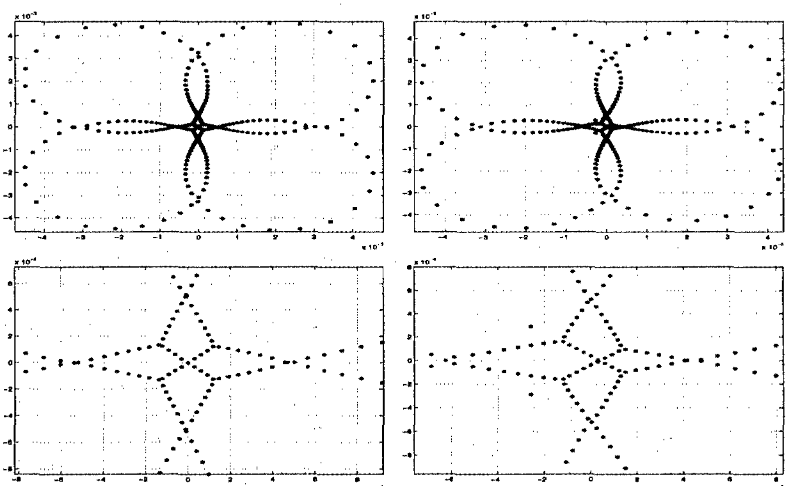

FIG. 5. Left: Case 6. Right: Case 7 and (below) zoomed versions

5 Conclusions

We have seen that it is easy to extend the detection of small solutions by numerical methods from one-dimensional to two-dimensional problems where the eigenvalues are real. Initial experiments indicate that the method works also for problems possessing complex eigenvalues, but here the patterns that arise in the eigenspecra plots are unfamiliar and require further investigation. However, based on our experimental evidence, it seems that small solutions arise in the latter case if and only if the eigenspectra plots pass through the origin.

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